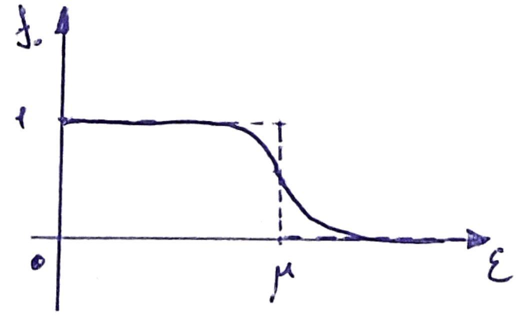
Electrical conductivity in metals

Equillibrium distribution function for electrons:

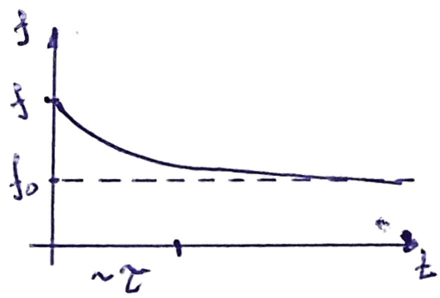
$$f_0 = \frac{1}{e^{\frac{\varepsilon-\mu}{kT}}+1} \quad \leftarrow \text{Fermi-Dirac distribution}$$



Let us apply external distortion (electric field).
Let us assume, that the distribution function can still be written as

$$f = \frac{1}{e^{\frac{\varepsilon(\vec{r})-\mu}{kT}}+1}.$$

In the approximation of relaxation time, $\frac{\partial f}{\partial t} = -\frac{f-f_0}{\tau}$.



$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \varepsilon}\cdot\frac{\partial \varepsilon}{\partial t} = \frac{\partial f}{\partial \varepsilon}\cdot\frac{\partial \varepsilon}{\partial \vec{k}}\cdot\frac{\partial \vec{k}}{\partial t}.$$

Equation of motion for an electron: $\hbar\cdot\frac{\partial \vec{k}}{\partial t} = e\vec{E}$

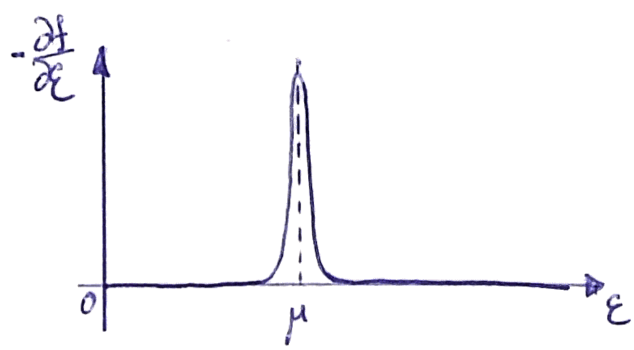
Group velocity of an electron: $\frac{1}{\hbar}\frac{\partial \varepsilon}{\partial \vec{k}} = \vec{v}$

$$-\frac{f-f_0}{\tau} = \frac{\partial f}{\partial \varepsilon}\cdot\vec{v}\cdot e\vec{E} \Rightarrow f = f_0 - \tau\cdot\left(\frac{\partial f}{\partial \varepsilon}\right)\cdot e\cdot\vec{v}\cdot\vec{E}$$

Induced electric current: $\vec{j} = \underset{\text{spin}}{2}\, e \sum_{\underset{\text{zone}}{n},\,\underset{\text{wavevector}}{\vec{k}}} f_{nk}\cdot\vec{v}$

Equillibrium part (f_0) does not contribute to the current;

$$\vec{j} = 2\tau e^2 \sum_{n,\vec{k}} \left(-\frac{\partial f}{\partial \varepsilon}\right)\vec{v}\,(\vec{v}\vec{E})$$



For an isotropic metal, $(\vec{v}\vec{E}) = \frac{1}{3}vE$

$$j = \tau e^2 \int \underbrace{-\left(\frac{\partial f}{\partial \varepsilon}\right)}_{\sim\delta(\varepsilon-\mu)}\cdot v\cdot\frac{1}{3}v\cdot \underset{\text{density of states (with spins)}}{D(\varepsilon)}\cdot d\varepsilon\cdot E$$

$$= \frac{1}{3}\tau e^2\left[v^2\cdot D(\varepsilon)\right]_{\varepsilon=\mu}\cdot E$$

$\leftarrow$ at the Fermi's surface

$$\sigma = \frac{1}{3}\tau e^2\left[v^2\cdot D(\varepsilon)\right] \leftarrow \text{electrical conductivity}$$

Thermal conductivity in metals (electronic part)

Now let us consider non-uniform distribution of temperature:

$$f = \frac{1}{e^{\frac{\varepsilon-\mu}{kT(\vec{r})}}+1}.$$

Denoting $\frac{\varepsilon-\mu}{kT} = z \Rightarrow$

$$\frac{\partial f}{\partial \varepsilon} = \frac{\partial f}{\partial z}\cdot\frac{1}{kT}$$

$$\frac{\partial f}{\partial T} = \frac{\partial f}{\partial z}\cdot -\frac{(\varepsilon-\mu)}{kT^2} = -\frac{\partial f}{\partial \varepsilon}\cdot\frac{\varepsilon-\mu}{T}$$

$$-\frac{f-f_0}{\tau} = \frac{\partial f}{\partial t} = \frac{\partial f}{\partial T}\cdot\underbrace{\frac{\partial T}{\partial \vec{r}}}_{\nabla T}\cdot\underbrace{\frac{\partial \vec{r}}{\partial t}}_{\vec{v}} = -\frac{\partial f}{\partial \varepsilon}\cdot\frac{\varepsilon-\mu}{T}\cdot(\nabla T)\cdot\vec{v} \Rightarrow f = f_0 + \tau\left(\frac{\partial f}{\partial \varepsilon}\right)\cdot\frac{\varepsilon-\mu}{T}\cdot\vec{v}\cdot(-\nabla T)$$

Heat: $dQ = TdS = dE - \mu dN$

Heat flow: $\vec{q} = 2\cdot\sum_{n,\vec{k}}(\varepsilon-\mu)\cdot f_{nk}\cdot\vec{v}$

Equillibrium part does not contribute to the heat flow:

$$\vec{q} = 2\cdot\tau\cdot\sum_{n,\vec{k}}\left(-\frac{\partial f}{\partial \varepsilon}\right)\cdot\frac{(\varepsilon-\mu)^2}{T}\cdot\vec{v}\cdot(\vec{v}\cdot(-\nabla T))$$

For an isotropic metal:

$$\vec{q} = \frac{1}{3}\tau\cdot\int_0^{+\infty}\left(-\frac{\partial f}{\partial \varepsilon}\right)\cdot\frac{(\varepsilon-\mu)^2}{T}\cdot v^2\cdot D(\varepsilon)\cdot d\varepsilon\cdot(-\nabla T) =$$

$$= \frac{1}{3}\tau\cdot\left[v^2\cdot D(\varepsilon)\right]_{\varepsilon=\mu}\cdot\underbrace{\int_0^{+\infty}\left(-\frac{\partial f}{\partial \varepsilon}\right)\cdot\frac{(\varepsilon-\mu)^2}{T}\cdot d\varepsilon}_{I = k^2T\cdot\frac{\pi^2}{3}}\cdot(-\nabla T)$$

$$\varkappa = \frac{\pi^2 k^2 T}{9}\cdot\tau\left[v^2\cdot D(\varepsilon)\right]_{\varepsilon=\mu}$$ – thermal conductivity ($\vec{q} = -\varkappa\nabla T$)

Wiedemann-Franz law: $\frac{\varkappa}{\sigma} = \frac{\pi^2}{3}\cdot\left(\frac{k}{e}\right)^2\cdot T$ ← does not depend on metal.

Evaluation of the integral $I = \int_0^{+\infty} \left(-\frac{\partial f}{\partial \varepsilon}\right) \cdot \frac{(\varepsilon-\mu)^2}{T} \cdot d\varepsilon$

$$I = \int_0^{+\infty} \frac{1}{\left(e^{\frac{\varepsilon-\mu}{kT}}+1\right)^2} \cdot \frac{e^{\frac{\varepsilon-\mu}{kT}}}{kT} \cdot \frac{(\varepsilon-\mu)^2}{T} \cdot d\varepsilon = k^2 T \cdot \int_{-\infty}^{+\infty} \frac{z^2 e^z dz}{(e^z+1)^2}.$$

$$\int_{-\infty}^{+\infty} \frac{x^2 e^x dx}{(e^x+1)^2} = \int_{-\infty}^{+\infty} \frac{x^2 dx}{(e^x+1)(e^{-x}+1)} = 2\int_0^{+\infty} \frac{x^2 dx}{(e^x+1)(e^{-x}+1)} = 2\int_0^{+\infty} \frac{x^2 e^{-x} dx}{(1+e^{-x})^2} =$$

$$\frac{1}{(1+e^{-x})^2} = 1 - 2e^{-x} + 3e^{-2x} - 4e^{-3x} + \dots$$

$$= 2 \cdot \int_0^{+\infty} x^2 \left(e^{-x} - 2e^{-2x} + 3e^{-3x} - 4e^{-4x} + \dots\right) dx$$

$$= 2 \cdot 2\left(1 - \frac{2}{2^3} + \frac{3}{3^3} - \frac{4}{4^3} + \dots\right) = 4\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right) = 4 \cdot \frac{\pi^2}{12} = \frac{\pi^2}{3}$$

Thermoelectric phenomena

One can consider simultaneous gradient of temperature and electric field:

$$\begin{cases} \vec{j} = L^{11}\vec{E} + L^{12}(-\nabla T) \\ \vec{q} = L^{21}\vec{E} + L^{22}(-\nabla T) \end{cases}$$ where L^{ij} - thermoelectric matrix.

It can be shown that $L^{11} = \sigma$ - electric conductivity.

$$L^{22} = \frac{\pi^2}{3}\left(\frac{k}{e}\right)^2 T \cdot \sigma$$

$$L^{21} = T L^{12}$$

In experiment, it is easier to control current $\vec{j}$, so the system can be rewritten:

$$\begin{cases} \vec{E} = \rho \cdot \vec{j} - S \cdot \nabla T \\ \vec{q} = \Pi \cdot \vec{j} - \varkappa \nabla T \end{cases}$$

ρ - resistivity

S - Seebeck coefficient $\left(S = -\frac{L^{12}}{L^{11}}\right)$

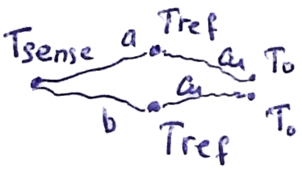
$\Pi = -T \cdot S$ - Peltier coefficient

$\varkappa = L^{22} - \frac{L^{12}L^{21}}{L^{11}} \approx L^{22}$ - thermal conductivity.

Seebeck effect: electric field in a non-uniformely heated metal.

$$\vec{E} = -\nabla\varphi = -S\nabla T \Rightarrow S = \frac{d\varphi}{dT}$$

Thermocouple

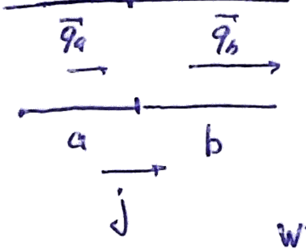


$$\Delta\varphi = \int_{T_0}^{T_{ref}} S_{Cu}\,dT + \int_{T_{ref}}^{T_{sens}} S_a\,dT + \int_{T_{sens}}^{T_{ref}} S_b\,dT + \int_{T_{ref}}^{T_0} S_{Cu}\,dT = \int_{T_{ref}}^{T_{sens}} (S_a - S_b)\cdot dT$$

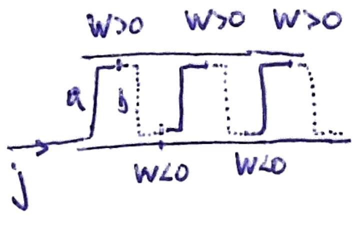
$$T_{sens} \sim T_{ref} + \frac{\Delta\varphi}{S_a - S_b}$$, sensitivity $\sim 10\text{-}100\ \frac{\mu V}{^\circ C}$

Peltier effect: heat is realeazed or absorbed at the contact, when current flows through the contact.

Peltier element:



$W = (\Pi_a - \Pi_b)j$ ← heat is released or absorbed at the contact depending on the current direction.



A Peltier element on semiconductors can reach $\Delta T \lesssim 70^\circ C$.